

# On Partitions of Goldbach's Conjecture

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An approximate formula for the partitions of Goldbach's Conjecture is derived using Prime Number Theorem and a probabilistic approach. A strong form of Goldbach's conjecture follows in the form of a lower bounding function for the partitions of Goldbach's conjecture. Numerical computations suggest that the lower and upper bounding functions for the partitions satisfy a simple functional equation. Assuming that this invariant scaling property holds for all even integer  $n$ , the lower and upper bounds can be expressed as simple exponentials.

## 1 Goldbach's Conjecture and Recent Progress

Goldbach's Conjecture states that every even integer  $> 2$  can be expressed as a sum of two primes.

The proof remains an unsolved problem since Goldbach first wrote the conjecture in a letter to Euler in 1792. However, significant progress has been made in recent years.

On the front of verifying Goldbach's Conjecture, no counter-example has been found to date. In 1855, Desboves [1] verified Goldbach's Conjecture for  $n < 10,000$ . In 1940, Pipping [2] verified the conjecture for  $n < 100,000$  by the use of a computer. In 1964, Stein & Stein [3, 4] verified the conjecture for  $n < 10^8$ . In 1989, Granville, d. Lune, te Riele [5] verified the conjecture for  $n < 2 \times 10^{10}$ . In 1998, Deshouillers, te Riele, Saouter [6] verified the conjecture for  $n < 10^{14}$ , and Richstein [7] verified the conjecture for  $n < 4 \times 10^{14}$ .

On the front of proving *weaker forms* of Goldbach's Conjecture, many have contributed. In 1930, Shnirelman [8] achieved the first breakthrough by proving that every natural number can be expressed as the sum of not more than 20 primes. In 1937, Vinogradov [9] proved that every sufficiently large odd numbers  $> N_1$  can be expressed as a sum of three primes. In 1966, Chen [10] proved that every sufficiently large even natural number is the sum of a prime and a product of at most two primes. In 1995, Ramaré

[11, 12] proved that every even integer can be expressed as the sum of six or fewer primes. In the same year, Kaniecki [13] proved that, assuming the Riemann hypothesis is true, every odd integer can be expressed as the sum of at most five primes. Kaniecki's proof can be improved to at most four primes pending a further computational verification.

On the front of studying the properties of the partitions of Goldbach's Conjecture, many asymptotic formulae have been derived by Vinogradov [9], Chen [10, 14] and others for partitions of the weaker forms of Goldbach's Conjecture but none has been derived for partitions of Goldbach's Conjecture. In 1923, Hardy & Littlewood [15] conjectured an asymptotic formula for the partitions of Goldbach's Conjecture. The asymptotic formula still remains to be proved to date.

In this paper, an approximate formula for the partitions of Goldbach's Conjecture is derived using Prime Number Theorem and a probabilistic approach. In contrast to the many weaker forms of Goldbach's Conjecture, a strong form of Goldbach's conjecture follows in the form of a lower bounding function for the partitions of Goldbach's conjecture. The lower bounding function can be estimated by the approximate formula which involves a sum of the products of the reciprocals of logarithms.

Numerical computations suggest that the lower and upper bounding functions for the partitions satisfy a simple functional equation. Assuming that this invariant scaling property holds for all even integer  $n$ , the lower and upper bounds can be expressed as simple exponentials.

## 2 Definitions

**Definition 2.1** *Partition function for Goldbach's Conjecture,  $G(n)$ , is defined as the number of representations of an even integer  $n$  as the sum of two primes  $p$  and  $q$ .*

$$G(n) = \# \{(p, q) \mid n = p + q, p \leq q\}.$$

**Proposition 1** *Hardy and Littlewood [15] gave a **conjectural** asymptotic formula,*

$$\lim_{n \rightarrow \infty} \frac{G(n)}{\int_2^n \frac{1}{(\ln x)^2} dx \prod_{\substack{k=2 \\ p_k | n}} \frac{p_k - 1}{p_k - 2}} = 2 C \quad (1)$$

where Hardy-Littlewood constant,

$$C = \prod_{k=2}^{\infty} \frac{p_k(p_k - 2)}{(p_k - 1)^2} = 0.6601618158. \quad (2)$$

**Theorem 1 (Prime Number Theorem)** *Let  $\pi(n)$  be the number of primes  $2 \leq p \leq n$ .*

$$\lim_{n \rightarrow \infty} \frac{\pi(n)}{n/\ln n} = 1. \quad (3)$$

**Theorem 2 (Chebyshev's Limits)** *Chebyshev [16] established the limits such that*

$$\frac{7}{8} < \frac{\pi(n)}{n/\ln n} < \frac{9}{8}. \quad (4)$$

### 3 Approximating $G(n)$ to first order

The following theorem is derived based on Prime Number Theorem and a heuristic probabilistic approach.

**Theorem 3**

$$G(n) \approx \sum_{k=3}^{n/2} \frac{1}{\ln(k) \ln(n-k)}, \quad \text{even } n \geq 6. \quad (5)$$

*Proof.* From Theorem 1 and 2, the Gauss's estimate  $\pi(n) \approx n/\ln n$  is obtained. The probability of  $n$  being a prime,

$$P(n) \approx \frac{\pi(n) - \pi(n-1)}{n - (n-1)} = \pi(n) - \pi(n-1) \quad (6)$$

$$\approx \left. \frac{d}{dx} \frac{x}{\ln x} \right|_{x=n} = \frac{1}{\ln n} - \frac{1}{(\ln n)^2} \approx \frac{1}{\ln n} \quad (7)$$

Pick an integer  $3 \leq k \leq n/2$ , even  $n \geq 6$ . The probability of  $k$  and  $n-k$  both being primes is  $P(k)P(n-k)$ . Since the partition function  $G(n)$  counts the number of pairs of primes,  $G(n)$  can be approximated by the sum of the probabilities  $P(k)P(n-k)$  over all  $k$  such that  $3 \leq k \leq n/2$ .

$$\begin{aligned}
G(n) &\approx \sum_{k=3}^{n/2} P(k) P(n-k), \quad \text{even } n \geq 6 & (8) \\
&\approx \sum_{k=3}^{n/2} \frac{1}{\ln(k) \ln(n-k)}
\end{aligned}$$

Q.E.D.

While (1) is a more accurate approximation than (5), computation of (1) requires the knowledge of the values of all primes  $p \mid n$ . (5) is a first order approximation of  $G(n)$  with no reference to the values of primes.

## 4 A Strong Form of Goldbach's Conjecture

From (7),

$$P(n) > \frac{1}{\ln n} - \left| \mathcal{O}\left(\frac{1}{(\ln n)^2}\right) \right| \quad (9)$$

It follows from (8) that (5) is approximating the lower bound of  $G(n)$ . This leads to a conjecture stronger than Goldbach's Conjecture where  $G(n) > 0$  for even  $n \geq 6$ .

### Conjecture 1 (A Strong Form of Goldbach's Conjecture)

$$\begin{aligned}
G(n) &> \sum_{k=3}^{n/2} \frac{1}{\ln(k) \ln(n-k)} \\
&\quad - \left| \mathcal{O}\left(\sum_{k=3}^{n/2} \frac{1}{[\ln(k)] [\ln(n-k)]^2}\right) \right| \\
&\quad - \left| \mathcal{O}\left(\sum_{k=3}^{n/2} \frac{1}{[\ln(k)]^2 [\ln(n-k)]}\right) \right| \\
&\quad (\text{even } n \geq 6) \\
&> 0. & (10)
\end{aligned}$$

## 5 Lower and Upper Bounding Functions of $G(n)$

**Definition 5.1** *The lower and upper bounding functions, respectively  $f_L(x)$  and  $f_U(x)$ , of  $G(n)$  are defined as respectively monotonous analytic functions such that  $f_L(n) < G(n) < f_U(n)$  for even integer  $n \geq 6$ .*

Numerical computations for  $n < 1,000,000$  suggest that  $f_L(x)$  and  $f_U(x)$  satisfy a functional equation of the form

$$\frac{f(ax)}{f(a)} = \frac{f(bx)}{f(b)}, \text{ constant } a, b > 0, x > 0. \quad (11)$$

A solution of the functional equation is

$$e^{\alpha x^\beta}, \text{ constant } \alpha > 0, 0 < \beta < 1. \quad (12)$$

This leads to the following conjecture.

**Conjecture 2** *The lower and upper bounding functions of  $G(n)$  can be expressed respectively as simple exponentials of the form  $\exp(\alpha x^\beta)$ , where constant  $\alpha > 0$ ,  $0 < \beta < 1$  can be determined by numerical computations.*

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